Journal of the Association of Arab Universities for Basic and Applied Sciences (2017) **24**, 254–262

University of Bahrain

Journal of the Association of Arab Universities for Basic and Applied Sciences

www.elsevier.com/locate/jaaubas
www.sciencedirect.com

ORIGINAL ARTICLE

Semi-analytical method for solving Fokker-Planck's equations

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Received 24 December 2016; revised 25 March 2017; accepted 14 July 2017
Available online 8 September 2017

KEYWORDS

Fokker-Planck's equation;
Semi-analytical method;
Analytic solution;
Brownian motion;
Kolmogorov's equation

Abstract In this paper, the linear and nonlinear Fokker-Planck equations (FPE) are solved by a semi-analytical iterative technique. This technique was proposed by Temimi and Ansari (TAM) in 2011. It is used to obtain the exact solutions for the 1D, 2D and 3D FPE. We solve several linear and nonlinear examples to show that the method is efficient and applicable. The results demonstrate that the presented method is very effective and reliable and does not require any restrictive assumptions for nonlinear terms. A symbolic manipulator Mathematica®10 was used to evaluate terms in the iterative process.

1. Introduction

There are many varieties of physical problems that are modeled by either ordinary or partial differential linear or nonlinear equations. Therefore, there is always a demand to develop reliable and efficient methods to obtain analytic solutions. During previous decades, scientists, mathematicians and physicists devoted considerable efforts to find either exact or numerical solutions for many nonlinear differential equations (ordinary or partial). Many methods have been proposed. For example: Adomian decomposition method (ADM) (Adomian, 1983, 1986, 1994; Tatari et al., 2007; Elhanbaly and Abdou, 2006; Zhang and Liang, 2015; Akram and Aslam, 2016; Beran and Čelikovský, 2016), the variational iteration method (VIM) (He, 1999, 2000, 2007; Biazar et al., 2010; Chang, 2016; Mohyud-Din et al., 2017; Siddiqi and Iftikhar, 2015), Darboux transformation method (Gu, 1999), exp-function method (He and Wu, 2006; Xu, 2007), modified simple equation (MSE) method (Khan and Akbar, 2013, 2014a,b; Khan et al., 2013a,b; Akter and Akbar, 2015), (G′/G)-expansion method (Borhanifar and Abazari, 2011; Akbar et al., 2013; Naher et al., 2013; Alam and Akbar, 2013, 2015), *F*-expansion method (Wang et al., 2003, Wang and Zhou, 2003; Zhou et al., 2003, 2004; Islam et al., 2014), tanh function method (Malfliet, 1992; Parkes and Duffy, 1997; Fan, 2000; Yan and Zhang, 2001; Zayed et al., 2004; Abdusalam, 2005; Xie et al., 2005), Backlund transformation method (Rogers and Shadwich, 1982) and many others.

One of the most prominent differential equations is the Fokker-Planck equation (FPE), which was used to describe the Brownian motion of particles (Risken, 1989) by Fokker and Planck. The FPE is featured in natural sciences

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Peer review under responsibility of University of Bahrain.

http://dx.doi.org/10.1016/j.jaubas.2017.07.001
1815-3852

(in different fields), including chemical physics, solid state physics, theoretical biology, quantum optics and circuit theory. Many current studies feature the FPE right now (Askari and Adibi, 2015; Saravanan and Magesh, 2016; Carlini and Silva, 2016; Busani, 2017; Wang and Duan, 2016; Gaviraghi et al., 2017).

Since the implementation of the FPE is not restricted to systems near thermal equilibrium, the FPE may be also implemented to systems away from the thermal equilibrium (Risken, 1989). For example, the statistics of laser light can be described by using the FPE. In a superionic conductor under the impact of a strong external field, there are many ions and all of these ions are considered a system out of the thermal equilibrium. A simple model of that system can be described by the FPE. When using the time-dependent formulation, the FPE can also be used for describing the dynamics of systems and not only for the evaluation of stationary properties (Risken, 1989).

The following equation describes the distribution function $W(x,t)$ for the motion when a small particle of mass m is immersed in a fluid (Risken, 1989):

$$\frac{\partial W}{\partial t} = \gamma \frac{\partial W}{\partial v} + \gamma \frac{KT}{m} \frac{\partial^2 W}{\partial v^2}, \tag{1}$$

where t represents the time, v represents the velocity of the Brownian motion of the immersed particle, K and T are the constant of Boltzmann and the fluid temperature respectively and γ represents the constant of fraction.

The TAM semi-analytical iterative method has been proposed by H. Temimi and A. R. Ansari to solve and deal with numerous kinds of nonlinear problems (Temimi and Ansari, 2011). The TAM is used to evaluate and find the exact and approximate solutions for different problems, such as, the nonlinear ordinary differential equations (Temimi and Ansari, 2015), nonlinear second order multi-point boundary value problems (Temimi and Ansari, 2011), KdV equations (Ehsani et al., 2013), duffing equations (AL-Jawary and Al-Razaq, 2016) and chemistry problems (AL-Jawary and Raham, 2016). The results obtained using TAM feature a high convergence and indicate that this method is appropriate, accurate and time efficient.

In this paper, we develop TAM for solving the 1D, 2D and 3D linear and nonlinear Fokker-Planck equation. The original contribution of the paper is the development of the TAM for the solution of the linear and nonlinear Fokker-Planck equations. The paper has been organized as follows: in Section 2 the Fokker-Planck equation is introduced. In Section 3, we review the basic idea of TAM. In Section 4 examples of solving the FPE by the TAM are shown and finally, the conclusions are given in Section 5.

2. The Fokker-Planck equation

The generalized formula of the FPE for two independent variables x and t is given in the following form (Risken, 1989):

$$\frac{\partial u}{\partial t} = \left(-\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right) u(x,t), \tag{2}$$

with the following initial condition

$$u(x,0) = f(x), x \in \mathbb{R}. \tag{3}$$

Here $B(x) > 0$ represents the diffusion coefficient and $A(x)$ represents the drift coefficient. These coefficients are may be functions of x and t. In this case, Eq. (3) is rewritten in the following way:

$$\frac{\partial u}{\partial t} = \left(-\frac{\partial}{\partial x} A(x,t) + \frac{\partial^2}{\partial x} B(x,t) \right) u(x,t). \tag{4}$$

The Eq. (2) describes the motion of the concentration field $u(x,t)$. This equation considered mathematically linear second order partial differential equation, which is of parabolic type. Also, this equation called the forward Kolmogorov equation. The following formula describes the backward Kolmogorov equation (Risken, 1989):

$$\frac{\partial u}{\partial t} = -\left(A(x,t) \frac{\partial}{\partial x} + B(x,t) \frac{\partial^2}{\partial x^2} \right) u(x,t). \tag{5}$$

The next Eq. (6) represents a generalized formula for Eq. (2) for J variables $x_1, x_2, \ldots, x_J$:

$$\frac{\partial u}{\partial t} = \left(-\sum_{i=1}^{J} \frac{\partial}{\partial x_i} A_i(\mathrm{x}) + \sum_{i,j=1}^{J} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(\mathrm{x}) \right) u(\mathrm{x},t), \tag{6}$$

with the following initial condition

$$u(\mathrm{x},0) = f(\mathrm{x}), \mathrm{x} = (x_1, x_2, \ldots, x_J) \in \mathbb{R}^J. \tag{7}$$

In general, we observe that in Eq. (6) each of the drift vector A_i and the diffusion tensor $B_{i,j}$ depend on all J variables $x_1, x_2, \ldots, x_J$.

The most general form of the FPE is the nonlinear FPE, which also has very important applications in different areas such as surface physics, plasma physics, laser physics, bio physics, polymer physics, population dynamic, nonlinear hydrodynamics, engineering, pattern formation, psychology, neurosciences and marketing (Frank, 2004). The nonlinear FPE for one variable has the following form

$$\frac{\partial u}{\partial t} = \left(-\frac{\partial}{\partial x} A(x,t,u) + \frac{\partial^2}{\partial x^2} B(x,t,u) \right) u(x,t), \tag{8}$$

and the nonlinear FPE for J variables $x_1, x_2, \ldots, x_J$ is

$$\frac{\partial u}{\partial t} = \left(-\sum_{i=1}^{J} \frac{\partial}{\partial x_i} A_i(\mathrm{x},t,u) + \sum_{i,j=1}^{J} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(\mathrm{x},t,u) \right) u(\mathrm{x},t), \tag{9}$$

where $\mathrm{x} = (x_1, x_2, \ldots, x_J) \in \mathbb{R}^J$.

3. The basic idea of the semi-analytical method (TAM)

To review the TAM algorithm, consider the following general differential equation (Temimi and Ansari, 2011, 2015):

$$L(u(x)) + N(u(x)) + g(x) = 0, \tag{10}$$

with the following initial conditions: $I\left(u, \frac{d^k u}{dx^k}\right) = 0$.

Here $u(x)$ is the unknown function, x represents the independent variable, while L and N represent the linear and the nonlinear operators, respectively. $g(x)$ represents the inhomogeneous term, which is a known function and I is an initial operator for the problem. I depends on µ and perhaps on the derivatives of order k, where k is a natural number. L is the linear part of the differential Eq. (10) and represents the main requirement. Also, one can take some linear parts and add them to the nonlinear parts N, as needed.

The TAN algorithm can be described as follows:

First consider the initial approximate function $u_0(x)$, which is the solution for the following initial problem

$$L(u_0(x)) + g(x) = 0, \text{ with } I\left(u_0, \frac{d^k u_0}{dx^k}\right) = 0, \tag{11}$$

Second, to find the next iterative function $u_1(x)$, the following problem must be solved

$$L(u_1(x)) + N(u_0(x)) + g(x) = 0, \text{ with } I\left(u_1, \frac{d^k u_1}{dx^k}\right) = 0. \tag{12}$$

The next and all other iterations can be evaluated in the same way. To find the $n+1$ iterative function, one must solve the following problem

$$L(u_{n+1}(x)) + N(u_n(x)) + g(x) = 0, \text{ with } I\left(u_{n+1}, \frac{d^k u_{n+1}}{dx^k}\right) = 0, \tag{13}$$

It can be observed that this iterative procedure is reliable and effective. Also, each solution is a development of the previous iterative solution. When we increase the number of iterations, we will obtain a new solution that converges to the solution of the current problem. The conditions for convergence of TAM are presented in (Temimi and Ansari, 2015).

4. Solution of the Fokker-Planck equation by TAM

In this section, the TAM method will be derived to solve several examples of the 1D, 2D, 3D linear and nonlinear Fokker-Planck equations.

Example 1. (AL-Jawary, 2016; Tatari et al., 2007; Yildirim, 2010; Biazar et al., 2010; Hesam et al., 2012) Let us consider the following 1D linear FPE in Eq. (2) with this initial condition

$$u(x,0) = x, x \in \mathbb{R}, \tag{14}$$

with $A(x) = -1$ and $B(x) = 1$.

Solution:

By implementing the same steps as described in the previous section, we first begin by solving the following initial problem to find the initial approximation $u_0(x,t)$

$$\frac{\partial u_0}{\partial t} = 0, \tag{15}$$

with an initial condition, which is equal to (14)

$u_0(x,0) = x.$

By solving the problem defined in Eqs. (14) and (15), we have

$u_0(x,t) = x,$

and then the second iteration $u_1(x,t)$ can be obtained by evaluating the following problem:

$$\frac{\partial u_1}{\partial t} = \left(-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right)u_0(x,t), \tag{16}$$

with an initial condition that equals to (14)

$u_1(x,0) = x.$

By solving the problem (16) with the initial condition, we obtain:

$u_1(x,t) = t + x.$

Also, by the same procedure, other solutions can be generated from evaluating the problems in the general form

$$\frac{\partial u_{n+1}}{\partial t} = \left(-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right)u_n(x,t), n \geqslant 2, \tag{17}$$

with an initial condition that equals to (14)

$u_{n+1}(x,0) = x.$

Therefore, in iterative procedure, we obtain the exact solution

$u(x,t) = t + x.$

This solution is the same as the result obtained by the DJM (AL-Jawary, 2016), ADM (Tatari et al., 2007), HPM (Yildirim, 2010), VIM (Biazar et al., 2010), and DTM (Hesam et al., 2012).

Example 2. (AL-Jawary, 2016; Tatari et al., 2007; Yildirim, 2010; Biazar et al., 2010; Hesam et al., 2012) Let us take the following 1D linear FPE in Eq. (4) with the following initial condition

$$u(x,0) = \sinh x, \quad x \in \mathbb{R}, \tag{18}$$

with $A(x,t) = e^t \coth x \cosh x + e^t \sinh x - \coth x$ and $B(x,t) = e^t \cosh x$.

Solution:

Applying the same steps as in the previous example, we first begin by solving the following initial problem in order to find the initial approximation $u_0(x,t)$

$$\frac{\partial u_0}{\partial t} = 0, \tag{19}$$

with an initial condition, which is the same as in (18)

$u_0(x,0) = \sinh x,$

we get

$u_0(x,t) = e^t \cosh x.$

The next iteration $u_1(x,t)$ can be obtained by solving the following problem:

$$\frac{\partial u_1}{\partial t} = \left(-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right)u_0(x,t), \tag{20}$$

with an initial condition which is equal to (18)

$u_1(x,0) = \sinh x.$

Solving the problem (20) with the corresponding initial condition, we obtain:$u_1(x,t) = (1+t)\sinh x.$

The iteration $u_2(x,t)$ can be evaluated by solving the following problem:

$$\frac{\partial u_2}{\partial t} = \left(-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right)u_1(x,t), \tag{21}$$

with an initial condition that equals to (18)

$u_2(x,0) = \sinh x.$

Solving the problem (21) with the given initial condition, we have:

$$u_2(x,t) = \left(1 + t + \frac{t^2}{2!}\right) \sinh x.$$

The other iterative solutions can be generated in the same way from evaluating the next problems, which are in the general form

$$\frac{\partial u_{n+1}}{\partial t} = \left(-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right)u_n(x,t), n \geqslant 3, \tag{22}$$

with the initial condition that equals to (18)

$u_{n+1}(x,0) = \sinh x.$

Hence, in iterative steps we have

$$u_n(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right) \sinh x.$$

As $n \to +\infty$ the explicit form of the exact solution is

$u(x,t) = e^t \sinh x.$

and this formula is equal to the solutions, which are calculated by the DJM (AL-Jawary, 2016), ADM (Tatari et al., 2007), HPM (Yildirim, 2010), VIM (Biazar et al., 2010), and DTM (Hesam et al., 2012).

Example 3. (AL-Jawary, 2016; Tatari et al., 2007; Yildirim, 2010; Biazar et al., 2010; Hesam et al., 2012) Let us take the following 1D linear FPE in Eq. (5) with this initial condition

$$u(x,0) = x + 1, \quad x \in \mathbb{R}, \tag{23}$$

where $A(x,t) = -1 - x$ and $B(x,t) = x^2e^t$.

Solution:

Implementing the same steps as in the previous examples, we begin by solving the following initial problem in order to find the initial approximation $u_0(x,t)$

$$\frac{\partial u_0}{\partial t} = 0, \tag{24}$$

with the following initial condition, which is given in (23)

$u_0(x,0) = x + 1.$

Now, solving the initial problem (21) with the given initial condition, we get

$u_0(x,t) = x + 1,$

The next iteration $u_1(x,t)$ is obtained by evaluating this problem:

$$\frac{\partial u_1}{\partial t} = \left(-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right)u_0(x,t), \tag{25}$$

with an initial condition which is equal to (23)

$u_1(x,0) = x + 1.$

Solving the problem (25), we obtain:

$u_1(x,t) = 1 + t + x + tx.$

The iteration $u_2(x,t)$ can be obtained by solving this problem:

$$\frac{\partial u_2}{\partial t} = -\left(\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right)u_1(x,t), \tag{26}$$

with an initial condition that equals to (23)

$u_2(x,0) = x + 1.$

Evaluating the problem (26), we have:

$$u_2(x,t) = 1 + t + \frac{t^2}{2!} + x + tx + +t^2x.$$

Thus, the other solutions will be generated by the same way from solving the next problems which are of the general form

$$\frac{\partial u_{n+1}}{\partial t} = -\left(\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right)u_n(x,t), n \geqslant 3, \tag{27}$$

with the initial condition that equals to (23)

$u_{n+1}(x,0) = x + 1.$

Hence, in iterative steps, we have

$$u_n(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right) + x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right).$$

As $n \to +\infty$ the explicit form of the exact solution is

$u(x,t) = e^t(x + 1).$

The form above is equal to the results that have been obtained by the DJM (AL-Jawary, 2016), ADM (Tatari et al., 2007), HPM (Yildirim, 2010), VIM (Biazar et al., 2010), and DTM (Hesam et al., 2012).

Example 4. (AL-Jawary, 2016; Tatari et al., 2007; Yildirim, 2010; Biazar et al., 2010; Hesam et al., 2012) Let us consider the following 2D linear FPE in Eq. (6) with the following initial condition

$$u(x_1,x_2,0) = x_1, \mathrm{x} = (x_1,x_2) \in \mathbb{R}^2, \tag{33}$$

with each of the following

$A_1(x_1,x_2) = x_1,$
$A_2(x_1,x_2) = 5x_2,$
$B_{1,1}(x_1,x_2) = x_1^2,$
$B_{1,2}(x_1,x_2) = 1,$
$B_{2,1}(x_1,x_2) = 1,$
$B_{2,2}(x_1,x_2) = x_2^2,$

Solution:

Implementing the TAM algorithm, we first solve the following initial problem in order to find the initial approximation $u_0(x_1,x_2,t)$

$$\frac{\partial u_0}{\partial t} = 0, \tag{34}$$

with the following initial condition which is the same as (33)

$u_0(x_1,x_2,0) = x_1.$

By solving the initial problem (34) with the corresponding initial condition, we get$u_0(x_1,x_2,t) = x_1,$

The second iteration $u_1(x_1,x_2,t)$ is obtained by solving the following problem:

$$\frac{\partial u_1}{\partial t} = \left(-\frac{\partial}{\partial x_1}A(x_1,x_2) + \frac{\partial^2}{\partial x_1 \partial x_2}B(x_1,x_2)\right)u_0(x_1,x_2,t), \tag{35}$$

with an initial condition, which is equal to (33)

$u_1(x_1, x_2, 0) = x_1.$

Solving the problem (35), we obtain:

$u_1(x_1, x_2, t) = (1+t)x_1.$

The iteration $u_2(x_1, x_2, t)$ can be obtained by solving this problem:

$$\frac{\partial u_2}{\partial t} = -\left(\frac{\partial}{\partial x_1}A(x_1, x_2) + \frac{\partial^2}{\partial x_1 \partial x_2}B(x_1, x_2)\right)u_1(x_1, x_2, t), \quad (36)$$

with an initial condition that equals to (33)

$u_2(x_1, x_2, 0) = x_1.$

Evaluating the problem (36), we have:

$u_2(x_1, x_2, t) = \left(1 + t + \frac{t^2}{2!}\right)x_1.$

Finally, the other solutions are generated by the same way from solving the next problems, which are in the general form

$$\frac{\partial u_{n+1}}{\partial t} = -\left(\frac{\partial}{\partial x_1}A(x_1, x_2) + \frac{\partial^2}{\partial x_1 \partial x_2}B(x_1, x_2)\right)u_n(x_1, x_2, t), n \geqslant 3, \quad (37)$$

with the initial condition that equals to (33)

$u_{n+1}(x_1, x_2, 0) = x_1.$

Thus, in iterative way we will have

$$u_n(x_1, x_2, t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right)x_1.$$

As $n \to +\infty$ the explicit form for the exact solution will be

$u(x_1, x_2, t) = e^t x_1.$

This function is the same as the results that have been evaluated by the DJM (AL-Jawary, 2016), ADM (Tatari et al., 2007), HPM (Yildirim, 2010), VIM (Biazar et al., 2010), and DTM (Hesam et al., 2012).

Example 5. (AL-Jawary, 2016; Tatari et al., 2007; Yildirim, 2010; Biazar et al., 2010; Hesam et al., 2012) Let us consider the following 3D linear FPE in Eq. (6) with the following initial condition

$$u(\mathrm{x},0) = x_3, \quad \mathrm{x} = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad (38)$$

with each of the following

$$\begin{aligned} A_1(x_1, x_2, x_3) &= 2x_1, \\ A_2(x_1, x_2, x_3) &= 2x_2, \\ A_3(x_1, x_2, x_3) &= 2x_3, \\ B_{1,1}(x_1, x_2, x_3) &= x_1, \\ B_{1,2}(x_1, x_2, x_3) &= 1, \\ B_{1,3}(x_1, x_2, x_3) &= 1, \\ B_{2,1}(x_1, x_2, x_3) &= 1, \\ B_{2,2}(x_1, x_2, x_3) &= x_2, \\ B_{2,3}(x_1, x_2, x_3) &= 1, \end{aligned}$$

Solution:

Applying the TAM algorithm given in the previous examples, the following initial problem will be solved in order to find the initial approximation $u_0(x_1, x_2, x_3, t)$

$$\frac{\partial u_0}{\partial t} = 0, \quad (39)$$

with the following initial condition, which is the same as (38)

$u_0(x_1, x_2, x_3, 0) = x_3.$

By solving the initial problem (39), we get

$u_0(x_1, x_2, x_3, t) = x_3,$

The second iteration $u_1(x_1, x_2, x_3, t)$ resulted by solving this problem:

$$\frac{\partial u_1}{\partial t} = \left(-\frac{\partial}{\partial x_1}A(x_1, x_2, x_3) + \frac{\partial^2}{\partial x_1 \partial x_2}B(x_1, x_2, x_3)\right)u_0(x_1, x_2, x_3, t), \quad (40)$$

with an initial condition which is equal to (38)

$u_1(x_1, x_2, x_3, 0) = x_3.$

Solving the problem (40), we obtain:

$u_1(x_1, x_2, x_3, t) = (1+t)x_3.$

The iteration $u_2(x_1, x_2, x_3, t)$ can be obtained by solving this problem:

$$\frac{\partial u_2}{\partial t} = -\left(\frac{\partial}{\partial x_1}A(x_1, x_2, x_3) + \frac{\partial^2}{\partial x_1 \partial x_2}B(x_1, x_2, x_3)\right)u_1(x_1, x_2, x_3, t), \quad (41)$$

with an initial condition that equals to (38)

$u_2(x_1, x_2, x_3, 0) = x_3.$

Evaluating the problem (41), we have:

$u_2(x_1, x_2, x_3, t) = \left(1 + t + \frac{t^2}{2!}\right)x_3.$

Thus, the other solutions will be generated by the same way from solving the next problems, which are in the general form

$$\frac{\partial u_{n+1}}{\partial t} = -\left(\frac{\partial}{\partial x_1}A(x_1, x_2, x_3) + \frac{\partial^2}{\partial x_1 \partial x_2}B(x_1, x_2, x_3)\right) \times u_n(x_1, x_2, x_3, t), n \geqslant 3, \quad (42)$$

with the initial condition that equals to (38)

$u_{n+1}(x_1, x_2, x_3, 0) = x_3.$

Thus, in an iterative way we will have

$$u_n(x_1, x_2, x_3, t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right)x_3.$$

As $n \to +\infty$ the explicit form for the exact solution will be

$u(x_1, x_2, x_3, t) = e^t x_3.$

This formula is equal to the result that has been evaluated by the DJM (AL-Jawary, 2016), ADM (Tatari et al., 2007), HPM (Yildirim, 2010), VIM (Biazar et al., 2010), and DTM (Hesam et al., 2012).

Example 6. (AL-Jawary, 2016; Tatari et al., 2007; Yildirim, 2010; Biazar et al., 2010; Hesam et al., 2012) Let us take the following 1D nonlinear FPE in Eq. (8) with the following initial condition

$$u(x,0) = x^2, \quad x \in \mathbb{R}, \tag{43}$$

with each of $A(x,t) = 4\frac{u(x,t)}{x} - \frac{x}{3}$ and $B(x,t) = u(x,t)$.

Solution:

Applying the same steps as in the previous examples, we first begin by solving the following initial problem in order to find the initial approximation $u_0(x,t)$

$$\frac{\partial u_0}{\partial t} = 0, \tag{44}$$

with an initial condition, which is the same as (43)

$$u_0(x,0) = x^2.$$

Solving the initial problem (44), we get

$$u_0(x,t) = x^2.$$

The next iteration $u_1(x,t)$ is obtained by evaluating this problem:

$$\frac{\partial u_1}{\partial t} = \left(-\frac{\partial}{\partial x}A(x,t,u_0) + \frac{\partial^2}{\partial x^2}B(x,t,u_0)\right)u_0(x,t), \tag{45}$$

with an initial condition, which is equal to (43)

$$u_1(x,0) = x^2.$$

Solving the problem (45), we obtain:

$$u_1(x,t) = (1+t)x^2.$$

The iteration $u_2(x,t)$ can be obtained by solving this problem:

$$\frac{\partial u_2}{\partial t} = \left(-\frac{\partial}{\partial x}A(x,t,u_1) + \frac{\partial^2}{\partial x^2}B(x,t,u_1)\right)u_1(x,t), \tag{46}$$

with an initial condition that equals to (43)

$$u_2(x,0) = x^2.$$

Evaluating the problem (46), we have:

$$u_2(x,t) = \left(1 + t + \frac{t^2}{2!}\right)x^2.$$

and so on. The other solutions can be generated by the same way from evaluating the next problems which are in the general form

$$\frac{\partial u_{n+1}}{\partial t} = \left(-\frac{\partial}{\partial x}A(x,t,u_n) + \frac{\partial^2}{\partial x^2}B(x,t,u_n)\right)u_n(x,t), n \geqslant 3, \tag{47}$$

with the initial condition that equals to (43)

$$u_{n+1}(x,0) = x^2.$$

Hence, in iterative steps we have

$$u_n(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots\right)x^2.$$

As $n \to +\infty$ the explicit form of the exact solution is

$$u(x,t) = x^2 e^t.$$

This formula is the same result as was calculated by the DJM (AL-Jawary, 2016), ADM (Tatari et al., 2007), HPM (Yildirim, 2010), VIM (Biazar et al., 2010), and DTM (Hesam et al., 2012).

Example 7. (AL-Jawary, 2016; Tatari et al., 2007; Yildirim, 2010; Biazar et al., 2010; Hesam et al., 2012) Let us consider the following 2D nonlinear FPE in Eq. (9) with the following initial condition

$$u(x_1,x_2,0) = x_1^2, \quad \mathrm{x} = (x_1,x_2) \in \mathbb{R}^2, \tag{48}$$

with each of the following

$$\begin{aligned}
A_1(x_1,x_2) &= \tfrac{4}{x_1}u(x_1,x_2,t),\\
A_2(x_1,x_2) &= x_2,\\
B_{1,1}(x_1,x_2) &= u(x_1,x_2,t),\\
B_{1,2}(x_1,x_2) &= 1,\\
B_{2,1}(x_1,x_2) &= 1,\\
B_{2,2}(x_1,x_2) &= u(x_1,x_2,t),
\end{aligned}$$

Solution:

Implementing the steps of the TAM, solving the following initial problem to find the initial approximation $u_0(x_1,x_2,t)$

$$\frac{\partial u_0}{\partial t} = 0, \tag{49}$$

with the following initial condition, which is the same to (48)

$$u_0(x_1,x_2,0) = x_1^2.$$

Solving the initial problem (49), we get

$$u_0(x_1,x_2,t) = x_1^2,$$

The second iteration $u_1(x_1,x_2,t)$ is obtained by solving this problem:

$$\frac{\partial u_1}{\partial t} = \left(-\frac{\partial}{\partial x_1}A(x_1,x_2,u_0) + \frac{\partial^2}{\partial x_1 \partial x_2}B(x_1,x_2,u_0)\right)u_0(x_1,x_2,t), \tag{50}$$

with an initial condition which is equal to (48)

$$u_1(x_1,x_2,0) = x_1^2.$$

Solving the problem (50), we obtain:

$$u_1(x_1,x_2,t) = (1-t)x_1^2.$$

The iteration $u_2(x_1,x_2,t)$ can be obtained by solving this problem:

$$\frac{\partial u_2}{\partial t} = -\left(\frac{\partial}{\partial x_1}A(x_1,x_2,u_1) + \frac{\partial^2}{\partial x_1 \partial x_2}B(x_1,x_2,u_1)\right)u_1(x_1,x_2,t), \tag{51}$$

with an initial condition that equals to (48)

$$u_2(x_1,x_2,0) = x_1^2.$$

Evaluating the problem (51), we have:

$$u_2(x_1,x_2,t) = \left(1 - t + \frac{t^2}{2!}\right)x_1^2.$$

The other solutions will be generated by the same way from solving the next problems, which are in the general form

$$\frac{\partial u_{n+1}}{\partial t} = -\left(\frac{\partial}{\partial x_1}A(x_1,x_2,u_n) + \frac{\partial^2}{\partial x_1 \partial x_2}B(x_1,x_2,u_n)\right) \times u_n(x_1,x_2,t), n \geqslant 3, \tag{52}$$

with the initial condition that equals to (48)

$u_{n+1}(x_1,x_2,0)=x_1^2.$

Thus, in iterative way we will have

$$u_n(x_1,x_2,t)=\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\dots\right)x_1^2.$$

As $n\to+\infty$ the explicit form for the exact solution will be

$u(x_1,x_2,t)=x_1^2e^{-t}.$

The above formula is similar to the results that have been evaluated by the DJM (AL-Jawary, 2016), ADM (Tatari et al., 2007), HPM (Yildirim, 2010), VIM (Biazar et al., 2010), and DTM (Hesam et al., 2012).

Example 8. (AL-Jawary, 2016; Tatari et al., 2007; Yildirim, 2010; Biazar et al., 2010; Hesam et al., 2012) Let us consider the following 3D nonlinear FPE in Eq. (9) with the following initial condition

$$u(\mathrm{x},0)=(x_3-1)^2,\quad \mathrm{x}=(x_1,x_2,x_3)\in\mathbb{R}^3, \tag{53}$$

with each of the following

$$\begin{aligned}
&A_1(x_1,x_2,x_3)=-x_1,\\
&A_2(x_1,x_2,x_3)=u(x_1,x_2,x_3,t),\\
&A_3(x_1,x_2,x_3)=\tfrac{2}{(x_3-1)},\\
&B_{1,1}(x_1,x_2,x_3)=u(x_1,x_2,x_3,t),\\
&B_{1,2}(x_1,x_2,x_3)=1,\\
&B_{1,3}(x_1,x_2,x_3)=1,\\
&B_{2,1}(x_1,x_2,x_3)=1,\\
&B_{2,2}(x_1,x_2,x_3)=u(x_1,x_2,x_3,t),\\
&B_{2,3}(x_1,x_2,x_3)=1,\\
&B_{3,1}(x_1,x_2,x_3)=1,\\
&B_{3,2}(x_1,x_2,x_3)=1,\\
&B_{3,3}(x_1,x_2,x_3)=1,
\end{aligned}$$

Solution:

Applying the TAM algorithm, we start by solving the initial problem to find the initial approximation $u_0(x_1,x_2,x_3,t)$

$$\frac{\partial u_0}{\partial t}=0, \tag{54}$$

with the following initial condition, which is the same as (53)

$u_0(x_1,x_2,x_3,0)=(x_3-1)^2.$

When the initial problem (54) will be solved, we will get

$u_0(x_1,x_2,x_3,t)=(x_3-1)^2,$

The second iteration $u_1(x_1,x_2,x_3,t)$ is obtained by solving this problem:

$$\frac{\partial u_1}{\partial t}=\left(-\frac{\partial}{\partial x_1}A(x_1,x_2,x_3,u_0)+\frac{\partial^2}{\partial x_1\partial x_2}B(x_1,x_2,x_3,u_0)\right)\times u_0(x_1,x_2,x_3,t), \tag{55}$$

with an initial condition, which is equal to (53)

$u_1(x_1,x_2,x_3,0)=(x_3-1)^2.$

Solving the problem (55), we obtain:

$u_1(x_1,x_2,x_3,t)=(1+t)(x_3-1)^2.$

The iteration $u_2(x_1,x_2,x_3,t)$ can be obtained by solving this problem:

$$\frac{\partial u_2}{\partial t}=-\left(\frac{\partial}{\partial x_1}A(x_1,x_2,x_3,u_1)+\frac{\partial^2}{\partial x_1\partial x_2}B(x_1,x_2,x_3,u_1)\right)\times u_1(x_1,x_2,x_3,t), \tag{56}$$

with an initial condition that equals to (53)

$u_2(x_1,x_2,x_3,0)=(x_3-1)^2.$

Evaluating the problem (56), we have:

$$u_2(x_1,x_2,x_3,t)=\left(1+t+\frac{t^2}{2!}\right)(x_3-1)^2.$$

Thus, the other solutions will be generated in the same way from solving the next problems, which are given in the general form as

$$\frac{\partial u_{n+1}}{\partial t}=-\left(\frac{\partial}{\partial x_1}A(x_1,x_2,x_3,u_n)+\frac{\partial^2}{\partial x_1\partial x_2}B(x_1,x_2,x_3,u_n)\right)\times u_n(x_1,x_2,x_3,t),n\geqslant 3, \tag{57}$$

with the initial condition that equals to (53)

$u_{n+1}(x_1,x_2,x_3,0)=(x_3-1)^2.$

Thus, in an iterative way we will have

$$u_n(x_1,x_2,x_3,t)=\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\dots\right)(x_3-1)^2.$$

As $n\to+\infty$ the explicit form for the exact solution will be

$u(x_1,x_2,x_3,t)=e^t(x_3-1)^2.$

This formula is the same as the results that have been evaluated by the DJM (AL-Jawary, 2016), ADM (Tatari et al., 2007), HPM (Yildirim, 2010), VIM (Biazar et al., 2010), and DTM (Hesam et al., 2012).

5. Conclusion

In this article, we develop an efficient semi-analytic method TAM for the solution of the Fokker-Planck equation. We prove the validity of our approach and show the efficiency of the proposed method by solving several test cases. The test cases include 1D, 2D, 3D linear and nonlinear implementations of the Fokker-Planck equations. All examples are solved successfully efficiently. The TAM is easy for applying and handling such kind of PDEs since implementation of this method does not require any restrictive assumptions for the nonlinear terms as required by some existing techniques such as the ADM, VIM and HPM. Also, the programming of this method is time-saver and economical in terms of computer processing and does not involve tedious evaluations. The test examples of FPE in this paper showed that the TAM is considered a reliable iterative method in dealing with such types of problems and providing the exact solutions.

Conflict of interest

There is no conflict of interest.

Acknowledgement

The authors would like to thank the anonymous referees, Managing Editor and Editor in Chief for their valuable suggestions.

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